# Multiple coverings of a closed set on a plane with non-Euclidean metrics by circles of two types

**Phung The Bao**[1,*] **and Le Quang Mung**[2]
*Tran Dai Nghia University, Vietnam*[1]
*Da Nang Department of Information and Communications, Vietnam*[2]

**ABSTRACT**
*The article is devoted to multiple covering problems for a closed set in a two-dimensional metric space by circles of two types. The number of circles of each class is given as well as a ratio of radii. This problem is as interesting and important as the classical circle covering problems. It is usually studied in the case when the distance between two points is Euclidean. We assume that the distance is determined using some particular metric arising in logistics, which, generally speaking, is not Euclidean. To solve this problem, we propose a computational algorithm based on a combination of the optical-geometric approach due to Fermat and Huygens principles and the Voronoi diagram. A key feature of the algorithm is the ability to deal with non-Euclidean metrics.*

***Keywords:*** *multiple covering, incongruent circles, non-Euclidean metric, Voronoi diagram, optical-geometric approach*

## 1. INTRODUCTION

The covering problems are widely used in various technical and economic fields of human activity. Examples of such tasks are locating ATMs, hospitals, schools, artificial Earth satellites, medical ambulance stations, cell towers [1-3], wireless sensors [4-6].

In general form, this problem is formulated as follows: how to locate geometric objects in a bounded area so that the covered area is completely inside in the union of these objects. Equal circles are often used as covering elements. In most cases, we are talking about the one-fold circle covering problem (CCP), which is considered in a large number of papers (for example, [7-9]).

There are other statements of the covering problem, such as the single covering with circles of different radii and the multiple covering with equal circles. In this case, as a rule, the radii ratio obeys the additional restrictions.

The problem of a single covering by unequal circles was first investigated by F. Toth and J. Molnar [10]. They proposed a hypothesis about the lower bound for the covering density. Then, this hypothesis was proven by G. Toth [11]. Florian and Heppes [12] established a sufficient condition for such a covering to be solid in the sense of [10]. Dorninger presented an analytical description for the general case (covering by unequal circles) in such a way that the conjecture can easily be numerically verified, and upper and lower limits for the asserted bound can be gained [13].

The multiple covering problem is as interesting and important as the classical CCP. Global navigation systems GPS (USA), Glonass (Russia), Baidu (China) and Galileo (EU) use a multiple covering (at least 3-fold) of the served areas to ensure positioning accuracy. For the multiple covering of a circle by congruent circles on a plane, the first exact results for k = 2, 3, 4 were obtained by Blundon [14]. Some analytical

Corresponding author: Dr. Phung The Bao
Email: phungthebao2003@gmail.com

results are obtained in the special cases when the covered area is a regular polygon [15-17]. These results are very important to verify the correctness of approximate results found by numerical methods. Among approximate methods, we can mention the greedy [18], heuristic [7, 19-20], and combinatorial [21] algorithms.

Note that the most of known results are obtained for the case when a covered set is a subset of the Euclidean space. In the case of a non-Euclidean metric, this problem is relatively poorly studied. Moreover, the problem of multiple covers with unequal circles, apparently, has not been considered yet.

In this paper, we deal with multiple covering by circles of two types with a specific non-Euclidean metric. This metric allows using the time as a measure of the distance [22-24]. We expand a technique based on the combination of the optical-geometric approach [22] and Dirichlet -Voronoi diagram [25-27]. The results of a computational experiment are presented and discussed.

## 2. PROBLEM STATEMENT AND MODELING

Let us consider some bounded field (service area) where it is required to locate a certain number of service facilities in such a way that their service zones, having a given shape, completely cover it. Such statements appear in problems of cell towers or security points placement [1, 28], designing energy-efficient monitoring of distributed objects by wireless sensor networks [5, 29], etc. The most straightforward problem statement of this type assumes that the service areas have the form of circles whose radii are the same, and it is enough to cover each point of the serviced space at least once. As a result, we have the classical circle covering the problem (see introduction). However, various complications and generalizations are possible in connection with applications.

Firstly, the need to take into account terrain features (for example, relief) leads to the fact that service areas cease to be circles. One way to solve this problem is to introduce a specific metric, which, in fact, replaces the physical distance between points by the minimum time it takes to pass the path between them [30-31].

Secondly, it is required that two or more objects service each point of the area. This situation is more typical for security tasks when it is necessary to ensure the correct operation of the system in case of failure of some of the servicing devices due to an accident or sabotage. However, such requirements may also apply to logistic systems (systems with duplication or redundancy).

Thirdly, service areas may be different. A similar requirement arises if we use service objects of various types.

Each of the additional requirements separately was previously considered (see, for example, [1, 5]). Moreover, we have already studied models in which two of the three conditions were taken into account simultaneously [23, 32]. However, three conditions are simultaneously considered for the first time. For definiteness, we will further talk about logistic systems (and serving logistic centers) and proceed to model designing.

We make a simplifying assumption. Suppose we are given a bounded domain in which consumers are continuously distributed, and there are only two types of logistic centers. Let $n$ and $m$ be a number of logistic centers of the first and second type, respectively, $\tau_1$ and $\tau_2$ be their maximum delivery time and $\tau_2 = \alpha\tau_1$; $\alpha > 0$. Here, the maximum delivery time is the time for which the goods are delivered to the most distant consumer at the border of the service area of the logistic centers means the "radius" of this zone. It is required to locate the centers so that each consumer must be serviced by at least $k$ of them ($k < n + m$), and the parameters

$\tau_1$; $\tau_2$ would be minimal.

Note that in logistics, such a statement is quite natural since the characteristics of the service centers directly affecting the delivery time of goods (such as the area of storage facilities, handling equipment, the capacity of parking lots and garages, etc.) are determined at the design stage. If we know only the total number of logistic centers $n + m$ and the multiplicity $k$, then the best placement is one with the shortest average delivery time $\tau = \tau_1(n + \alpha m) = (n + m)$.

Next, we turn to the mathematical formulation of the described problem.

## 3. MATHEMATICAL FORMULATION

Assume we are given a metric space $X$, a bounded domain $M \subset X$ with a continuous boundary $\partial M$, $n$ circles $C_i(O_i, R_1)$ and $m$ circles $C_i(O_i, R_2)$; here $O_i(x_i, y_i)$ is a circle centers, $R_1$ and $R_2$ are radii. Let $f(x, y) > 0$ be a continuous function, which shows the instantaneous speed of movement at every point of $X$. The minimum moving time between two points $a, b \in X$ is determined as follows:

$$\rho(a,b) = \min_{\Gamma \in G(a,b)} \int_{\Gamma} \frac{d\Gamma}{f(x,y)}, \qquad (1)$$

where $G(a,b)$ is the set of continuous curves, which belong to $X$ and connect two points, $a$ and $b$. It is easy to verify that for the distance determined by the formula (1), all metric axioms are satisfied. In logistic problems, in particular, $\rho(a, b)$ determines the minimum time for the delivery of goods between points. Still, it may also have another meaning, for example, determining the geodetic distance. Therefore, to avoid direct association with transportation, we will further use the traditional symbol $R$ to designate the circle radius in metric (1).

It is required to locate the circles to minimize the radii and to cover $M$ at least $k$ times. The last means that every point of $M$ must belong at least *k in* different circles.

In other words, we have the following optimization problem:

$$R_1 \to \min \qquad (2)$$

$$R_2 = \alpha R_1, \alpha \in R^+, \qquad (3)$$

$$\max_{j \in J_k(s)} \omega\rho\left(s, O_j\right) \le R_1 \qquad (4)$$

Here $\omega = \begin{cases} 1, i = 1,...,n \\ 1/\alpha, i = n+1,...,n+m \end{cases}$, and $J_k(s)$ is the set of indexes (numbers) of $k$ centers, that locate closer to $s$ than other $n + m - k$ centers:

$$J_k(s) = \left\{q_j, j = 1,...,k : \rho(O_{qj}, p) \le \omega\rho(O_l, p) \forall l = \{1,...,n+m\} \setminus \{q_1,...,q_j\}\right\}$$

The objective function (2) minimizes the radius of the covering. Constraint (3) fixes the radii ratio, and (4) guarantees that each point of $M$ belongs to at least $k$ circles.

Note, if $\alpha = 1$, we have the multiple covering of a bounded domain by equal circles with non-Euclidean metric [32].

## 4. SOLUTION METHOD

In this section, we propose a numerical method for solving t problem (2)-(4). We combine the analogy between the propagation of the light wave and finding the minimum of integral functional (1) and Dirichlet-Voronoi diagram.

The concept of *k-th* order Voronoi diagrams was introduced by F.L.Toth [27] and earlier was used in studies [15, 25, 32]. To apply it, at first, we should determine a $k$-fold Voronoi-Dirichlet region for the case of two types of circles.

For a set of $n + m$ points $O_i$, the generalized $k$-fold Voronoi region $M_i^k$ centered at $O_i$ is defined as follows:

$$M_i^k = \left\{p \in M : \rho\left(p, O_i\right) \le \max_{j \in J_k(p)} \lambda\rho\left(p, O_j\right)\right\}, i = 1,...,n+m, k < n+m, \qquad (5)$$

where $\lambda = \begin{cases} 1, i, j = 1,...n; i, j = n+1,...,n+m, \\ 1/\alpha, i = 1,...,n; j = n+1,...,n+m, \\ \alpha, i = n+1,...,n+m; j = 1,...,n. \end{cases}$

Figure 1 shows double Voronoi-Dirichlet regions (grey color) for the case of four circles (we point out their centers only) where the radii of circles 1 and 2 are equal, but they are three times larger than the radii of circles 3 and 4.

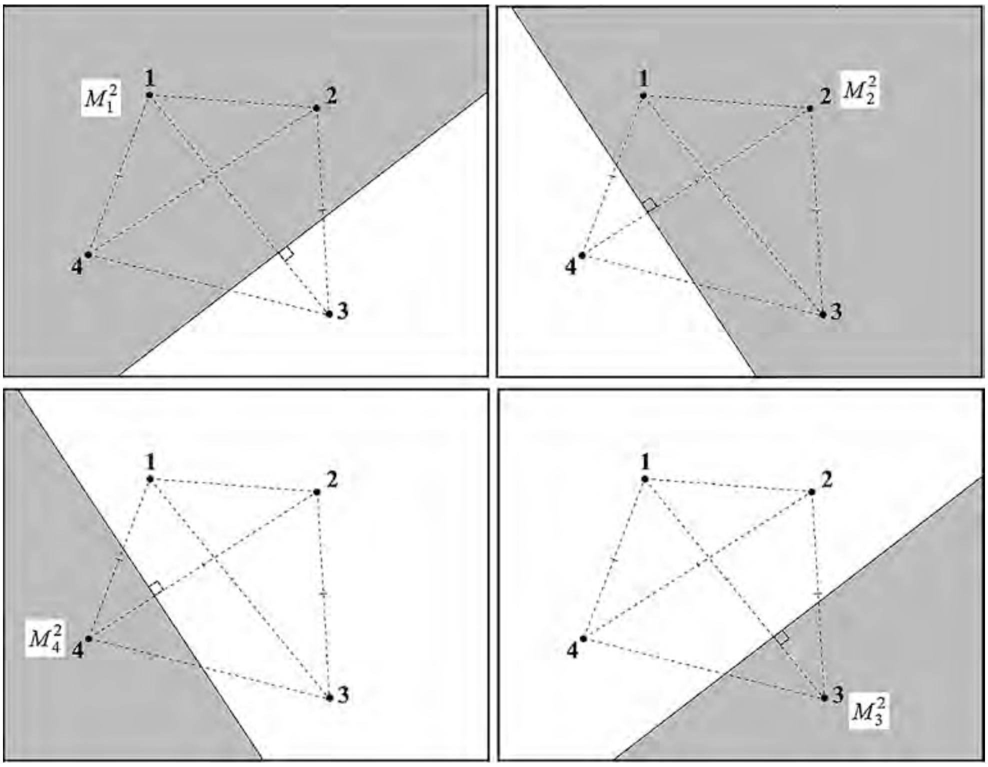


**Figure 1.** Double Voronoi-Dirichlet regions

At first, we propose the OCMC (One Covering Minimum Circle) algorithm, which allows finding a circle $C(O^*, R^*)$ centered in $O^*$, covering region *M* and having an approximately minimal radius $R^*$.

The principle of this algorithm is that the randomly generated center of the circle moves in the direction of decreasing the maximum distance from it to the boundary of the covered region. This process finishes when the coordinates of the center stop changing (Figure 2).

Here and further, we cover set $M$ by a uniform rectangular grid with the step $h$ and deal with set $M^h$ approximating $M$. For brevity, we omit the index *h*.

**Algorithm OCMC**

Step 1. Put $R^* = +\infty, Iter = 1$.

Step 2. Randomly generate initial coordinates of a point $O(x,y) \in M$.

Step 3. Define the set of nearest points for the point *O*:

$$\Delta O = \left\{ O(x+\chi, y+\sigma) \in M^h : \chi, \sigma \in \{-h, 0, h\} \right\}$$

Step 4. Find a point $O_{new}$:

$$O_{new} = \arg\min_{p \in \Delta O} \max_{s \in \partial M} \rho(p, s)$$

Step 5. If $\rho(O_{new}, \partial M) \le \rho(O, \partial M)$, then put $O := O_{new}$ and go to Step 3.

Step 6. If $\rho(O, \partial M) < R^*$, then put $O^* := O, R^* := \rho(O^*, \partial M)$.

Step 7. The counter Iterate of an initial solution generations is incremented. If it becomes equal a certain prescribed value, then the algorithm is terminated. Otherwise, go to Step 2.

The general algorithm includes the basic steps: constructing the generalized *k*-fold Voronoi diagram for the initial set of centers; moving $O_i$ to the point $O_i^*$, that is the center of the covering circle, which has the minimum radius for each part of the diagram; revising radius ratio and returning to the first step with the new centers. Now we describe the general algorithm in details.

**General algorithm**

Step 1. Randomly generate initial coordinates of the circles centers $O_i \in M, i=1,...,n+m$.

Step 2. From $O_i \in M, i=1,...,n+m$, we initiate the light waves using the algorithm from [22]. The speed of a light wave emitted from points $O_i, i=1,...,n$ is $\alpha$ times less than from $O_i$; $i = n + 1, ..., n + m$. This allows us to find the time $T_i(x, y)$; $i = 1, ..., n$ which is required to reach $s(x, y)$ by each wave. For every $s(x,y) \in M$ we obtain vector $T(x,y) = T_i(x,y)$.

Step 3. For each $s(x,y)$ we choose $k$ minimal components of vector $T(x,y)$. Thus, we obtain $J_k(s)$ which is the index set of Voronoi domains contained $s(x,y)$.

Step 4. Find $k$-fold Voronoi domain $M_i^k, i=1,...,n+m$ and their boundaries $\partial M_i^k$.

Step 5. For each $M_i^k, i=1,...,n+m$ we find a minimal covering circle $C_i(O_i^*, R_i^*)$ by OCMC algorithm.

Step 6. To ensure full covering of $M$ by circles, we choose the maximum radius $R_1 = \max_{i=1,...,n} R_i$ and $R_2 = \max_{i=n+1,...,n+m} R_i$.

Step 7. Check the inequality $R_2 \geq \alpha R_1$. If it is sa-tisfied, then put $R_1 = R_2/\alpha$, otherwise, put $R_2 = \alpha R_1$.

Step 8. If the value of the founded radius is less than the previous one, we save the current radius and the current set of circles. The counter of an initial solution generations is incremented. If it becomes equal a certain prescribed value, then the algorithm is terminated. Otherwise, go to Step 1.

A drawback of the algorithm is that it does not guarantee a solution that globally minimizes the circles radii. This feature is inherited from the constructing of Voronoi diagram. We use multiple generating of initial positions (Step 1) to increase the probability of finding a global solution.

## 5. COMPUTATIONAL EXPERIMENT

The algorithms are implemented in C# using the Visual Studio 2015. The numerical experiment was carried out using the PC of the following configuration: Intel (R) Core i5-3570K (3.4 GHz, 8 GB RAM) and Windows 10 operating system.

Note that in the tables, $n$ is a number of big circles, $m$ is a number of small circles, $k$ is multiplicity of the covering, $R_{n,m}^k$ is the best radius of the big circles, $\Delta R_{n,m}^k = \frac{n+\alpha m}{n+m} R_{n,m}^k$ is the average radius of the covering. In Fig. 4 and Fig. 6 the circles in each group here are equal.

In the figures, the origin is located in the upper left corner, the bold black closed curves are large circles, the thin ones are small circles, the grey dots are the centers of circles, the dashed line lines are the boundary of container $M$. The number of random generations $Itear$ = 100, the grid step $h = 0,001$.

*Example 1.* This example illustrates how the proposed in the previous section algorithm works in the case of the Euclidean metric $f(x, y) \equiv 1$. The covered set is a square with a side equals to $3, \alpha = 0,5, k = 2, 3, 4$. Table 1 shows the best solutions for 15 circles.

**Table 1.** The best coverings of a square by 15 circles with Euclidean metric

| $n$ | $m$ | $R_{n,m}^2$ | $\Delta R_{n,m}^2$ | $R_{n,m}^3$ | $\Delta R_{n,m}^3$ | $R_{n,m}^4$ | $\Delta R_{n,m}^4$ |
|---|---|---|---|---|---|---|---|
| 14 | 1 | 0.27739 | 0.26814 | 0.34132 | 0.32994 | 0.50000 | 0.48333 |
| 13 | 2 | 0.27877 | 0.26018 | 0.35184 | 0.32838 | 0.50000 | 0.46667 |
| 12 | 3 | 0.28985 | 0.26086 | 0.35358 | 0.31823 | 0.50018 | 0.45016 |
| 11 | 4 | 0.30170 | 0.26147 | 0.37174 | 0.32217 | 0.50028 | 0.43357 |
| 10 | 5 | 0.30699 | 0.25583 | 0.38873 | 0.32394 | 0.50584 | 0.42154 |
| 9 | 6 | 0.31457 | 0.25166 | 0.40089 | 0.32071 | 0.51499 | 0.41199 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 7 | 0.32299 | 0.24763 | 0.41846 | 0.32082 | 0.52389 | 0.40165 |
| 7 | 8 | 0.33483 | **0.24554** | 0.42964 | **0.31507** | 0.53729 | 0.39401 |
| 6 | 9 | 0.35668 | 0.24968 | 0.46228 | 0.32359 | 0.55318 | 0.38723 |
| 5 | 10 | 0.37642 | 0.25094 | 0.50071 | 0.33381 | 0.55607 | 0.37071 |
| 4 | 11 | 0,39016 | 0.24710 | 0.51579 | 0.32667 | 0,58426 | **0.37003** |
| 3 | 12 | 0.41236 | 0.24742 | 0.54103 | 0.32462 | 0.63004 | 0.37802 |
| 2 | 13 | 0.44312 | 0.25110 | 0.56356 | 0.31935 | 0.70711 | 0.40069 |
| 1 | 14 | 0.47796 | 0.25491 | 0.61036 | 0.32553 | 0.70711 | 0.37712 |

Table 1 shows that the radii of circles, as one would expect, grow with an increase in the number of small circles and a simultaneous decrease in the number of large ones. Other, more specific laws could not be identified. It is noteworthy that for a 4-fold covering, the radii $R^4_{2,13}$ and $R^4_{1,14}$ are equal.

Average radii behave even less regularly. The best 2,3-fold coverings consist of 7 large circles and 8 small ones (Figure 2), and 4-fold covering contains 4 large circles and 11 small ones. In addition, we note that the average radius of 2-fold covering with circles of two types is always less than the best radius of 2-fold covering with equal ones $R^1_{15}$=0.27012 (see [32]).

The operating time is 3’20’’ ÷ 4’34’’.

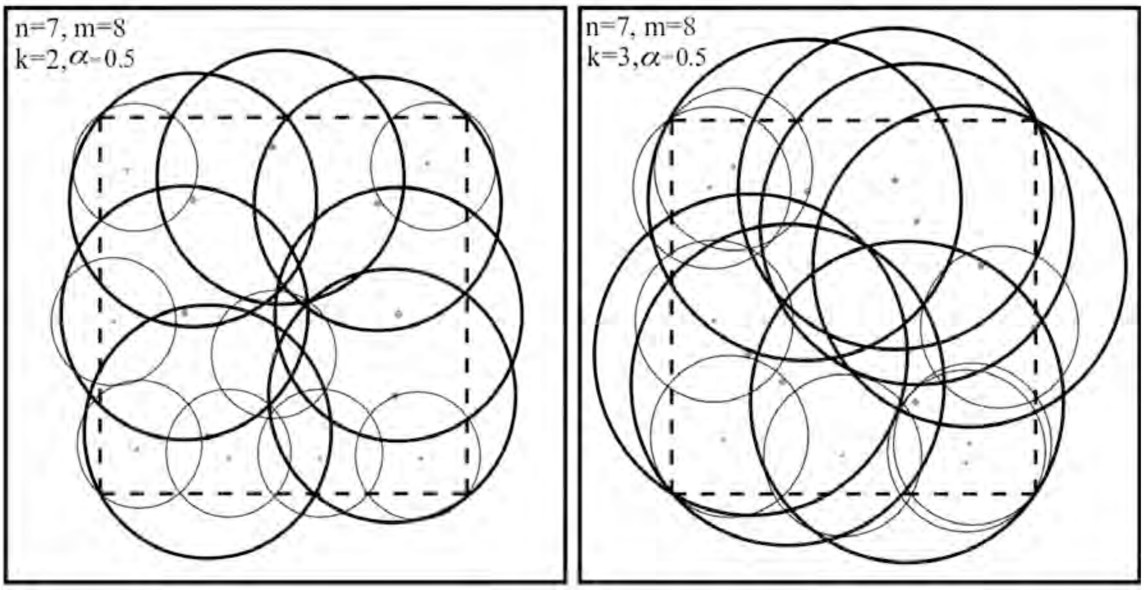


**Figure 2.** The best 2-fold (left) and 3-fold (right) coverings with 15 circles

*Example 2.* Let $f(x,y) = 0,5 + 2x$. It means that instantaneous speed of movement increases linearly along the coordinate *x*. The covered set *M* is following:

$$M = \left\{(x,y) : (x-2,5)^2 + (y-2,5)^2 \le 4\right\}$$

The best solutions for the cases of 2,3,4-fold coverings with 13 circles of two types are shown in Table 2. Here the radii ratio is 1/3.

Note that in this case the wave fronts also have the form of a circle, as in the Euclidean metric, but the source of the wave (the center of the circle) is displaced (see more in [33]). The apparent size of the covering circles depends on the location of their centers: the closer it to the axis $Oy$, the smaller it looks (Figure 3). We emphasize that in the given metric the radii are equal.

**Table 2.** The best coverings of a circle by 13 circles with the "linear" metric

| $n$ | $m$ | $R^2_{n,m}$ | $\Delta R^2_{n,m}$ | $R^3_{n,m}$ | $\Delta R^3_{n,m}$ | $R^4_{n,m}$ | $\Delta R^4_{n,m}$ |
|---|---|---|---|---|---|---|---|
| 12 | 1 | 0.27457 | 0.26049 | 0.35948 | 0.34105 | 0.43305 | 0.41084 |
| 11 | 2 | 0.28117 | 0.25233 | 0.37712 | 0.33844 | 0.45014 | 0.40397 |
| 10 | 3 | 0.29531 | 0.24988 | 0.39511 | 0.33433 | 0.46838 | 0.39632 |
| 9 | 4 | 0.30825 | 0.24502 | 0.41499 | 0.32987 | 0.47750 | 0.37955 |
| 8 | 5 | 0.32948 | 0.24499 | 0.43755 | 0.32536 | 0.48785 | 0.36276 |
| 7 | 6 | 0.35322 | 0.24454 | 0.45458 | 0.31471 | 0.49786 | 0.34467 |
| 6 | 7 | 0.37173 | 0.23829 | 0.48095 | 0.30830 | 0.50551 | 0.32404 |
| 5 | 8 | 0.39442 | 0.23261 | 0.48902 | 0.28839 | 0.51528 | 0.30388 |
| 4 | 9 | 0.43990 | 0.23687 | 0.49312 | 0.26553 | 0.52712 | 0.28383 |
| 3 | 10 | 0.46108 | 0.22463 | 0.50384 | 0.24546 | 0.53098 | 0.25868 |
| 2 | 11 | 0.49159 | 0.21428 | 0.52712 | 0.22977 | 0.57789 | 0**.25190** |
| 1 | 12 | 0.50774 | **0.19528** | 0.55580 | **0.21377** | 0.72432 | 0.27859 |

Table 2 shows that the radii of circles, as in the previous example, grow with an increase in the number of small circles. The average radii decrease monotonously with an increase in the number of small circles. The best 2,3-fold coverings consist of 1 large circle and 12 small ones, and 4-fold covering contains 2 large circles and 11 small ones.

Figure 3 (right) illustrates the interesting 3-fold covering. It splits into 1-fold covering by 1 large circle and 2-fold covering by 12 small ones.

The operating time of the proposed algorithm is 3'11'' ÷ 4'08''.

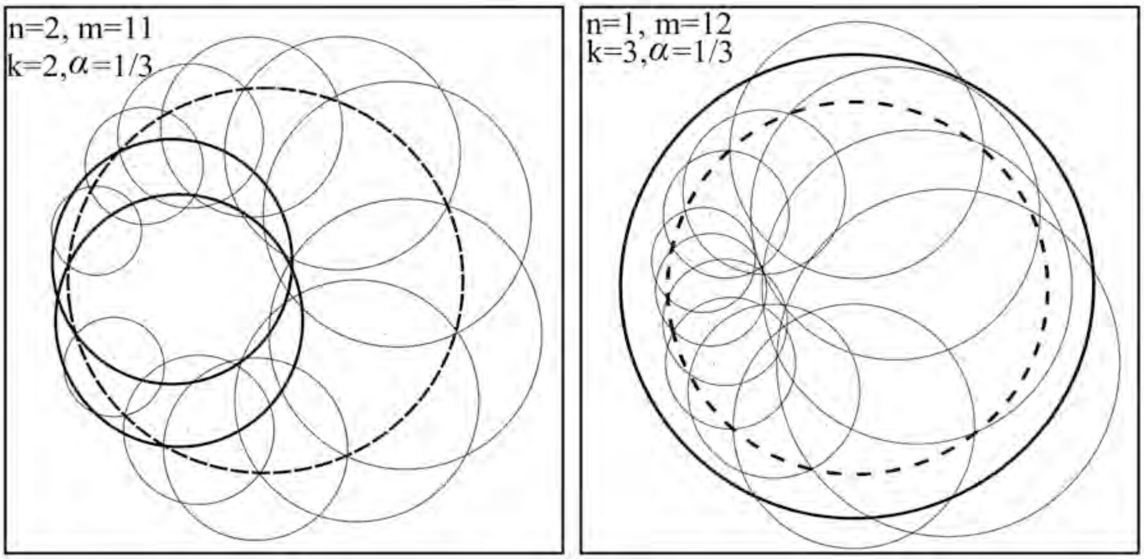


**Figure 3.** The best 2-fold (left) and 3-fold (right) coverings of a circle by 13 circles with the "linear" metric

*Example 3.* Let the covered set *M* is a polygon with the vertices:

{(0.5, 1.5), (0.5, 3.5), (1.5, 4.5), (3.5, 4.5), (4.5, 3.5), (4.5, 1.5), (3.5, 0.5), (1.5, 0.5)}.

The instantaneous speed of movement $f(x; y)$ is defined as follows:

$$f(x,y) = 1 + \frac{3}{(x-2)^2 + (y-2)^2 + 1}$$

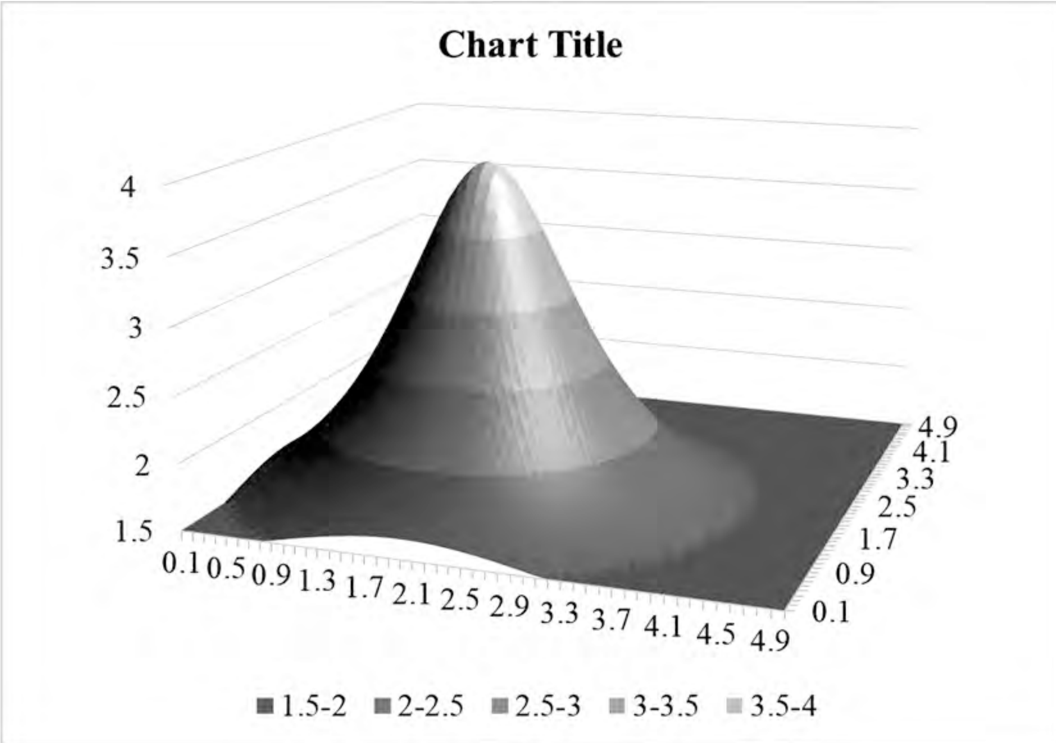


**Figure 4.** Level lines of function f(x; y)

Figure 4 shows level lines of $f(x; y)$. From the lowest to the highest point, the wave speed increases.

Table 3 shows the best coverings of $M$ by 19 circles of two types for $\alpha=1/4$ . One can see that the radii of the circles, as in the two previous examples, grow with an increasing number of small circles. Moreover, the increase in all cases occurs with acceleration.

The best 2-fold coverings consist of 7 large and 12 small circles, 3-fold coverings includes of 10 large and 9 small circles, and 4-fold covering contains 2 large circles and 17 small ones. Figure 6 shows that the wave fronts differ significantly from the circles, and the covering elements have an oviform shape.

The operating time is 4’40’’ ÷ 6’05’’.

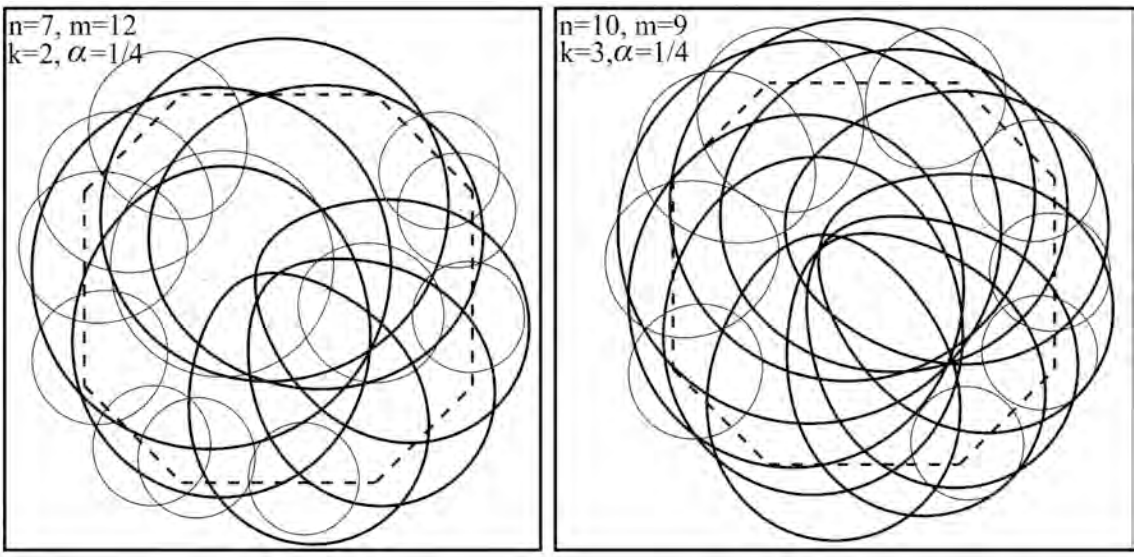


**Figure 5.** The best 2-fold (left) and 3-fold (right) coverings of a polygon by 19 circles with the non-Euclidean metric

**Table 3.** The best multiple coverings of a polygon by 19 circles with the non-Euclidean metric

| $n$ | $m$ | $R^2_{n,m}$ | $\Delta R^2_{n,m}$ | $R^3_{n,m}$ | $\Delta R^3_{n,m}$ | $R^4_{n,m}$ | $\Delta R^4_{n,m}$ |
|---|---|---|---|---|---|---|---|
| 18 | 1 | 0.51256 | 0.49232 | 0.63925 | 0.61401 | 0.78887 | 0.75773 |
| 17 | 2 | 0.53152 | 0.48956 | 0.65804 | 0.60609 | 0.81266 | 0.74851 |
| 16 | 3 | 0.55448 | 0.48882 | 0.69060 | 0.60882 | 0.85300 | 0.75199 |
| 15 | 4 | 0.56005 | 0.47162 | 0.71822 | 0.60482 | 0.86624 | 0.72947 |
| 14 | 5 | 0.58186 | 0.46702 | 0.74946 | 0.60154 | 0.91106 | 0.73125 |
| 13 | 6 | 0.60108 | 0.45872 | 0.77528 | 0.59166 | 0.96564 | 0.73693 |
| 12 | 7 | 0.62704 | 0.45378 | 0.80655 | 0.58369 | 0.96823 | 0.70069 |
| 11 | 8 | 0.64791 | 0.44331 | 0.82150 | 0.56208 | 1.04158 | 0.71266 |
| 10 | 9 | 0.68587 | 0.44221 | 0.85210 | **0.54938** | 1.07594 | 0.69370 |
| 9 | 10 | 0.71505 | 0.43280 | 0.94860 | 0.57416 | 1.11342 | 0.67391 |
| 8 | 11 | 0.76475 | 0.43269 | 0.98175 | 0.55546 | 1.14014 | 0.64508 |
| 7 | 12 | 0.80687 | **0.42467** | 1.05814 | 0.55692 | 1.19896 | 0.63103 |
| 6 | 13 | 0.87353 | 0.42527 | 1.18404 | 0.57644 | 1.27998 | 0.62315 |
| 5 | 14 | 0.96564 | 0.43200 | 1.25147 | 0.55987 | 1.38141 | 0.61800 |
| 4 | 15 | 1.05610 | 0.43078 | 1.36896 | 0.55839 | 1.45417 | 0.59315 |
| 3 | 16 | 1.16501 | 0.42921 | 1.50528 | 0.55458 | 1.62250 | 0.59776 |
| 2 | 17 | 1.29896 | 0.42729 | 1.67998 | 0.55262 | 1.74612 | **0.57438** |
| 1 | 18 | 1.47143 | 0.42594 | 1.90411 | 0.55119 | 2.02766 | 0.58695 |

## 6. CONCLUSION

The paper considers one of the topical problems for logistic and security systems: optimal placement of various service facilities (sensors, CCTV cameras, logistic centers) with the reservation (duplication). We formulate the subject problem in the form of the problem of constructing an optimal $k$-fold covering of a bounded set by circles of two types. At the same time, we use a specific non-Euclidean metric to take into account the local characteristics of the service area (for example, relief). The metric is determined by minimizing the integral functional of a function that defines the speed of movement. In other words, it replaces the physical distance between points by the minimum time it takes to pass the path between them. To solve the optimization problem, we suggest an original computational algorithm based on the combination of the optical-geometric approach and a new method for constructing generalized multiple Voronoi diagrams. We have already presented algorithms based on these principles [25, 32]; however, in this case, the procedure for constructing multiple Voronoi diagram is much more complicated. The reason is the presence of various types of elements in the covering, which in turn often leads to the non-convexity and the multiple connection of Voronoi regions. The algorithm is implemented, and a computational experiment is carried out. It shows that the developed tools effectively solve the problem with the number of objects up to 20. Besides, it turned out that in the best (from the application domain point of view) covering, as a rule, objects of both types are present. This fact is an additional confirmation of the relevance of the study. Further studies may be associated, firstly, with an increase in the number of types of covering elements; secondly, with an increase in the adequacy of the model, in particular, the use of two-level optimization problems as a mathematical formalization.

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# Bao phủ nhiều lần một vùng có kích thước cố định bằng hai loại đường tròn khác nhau

**Phùng Thế Bảo* và Lê Quang Mừng**

**TÓM TẮT**

*Bài báo nghiên cứu về bài toán bao phủ nhiều lần một vùng khép kín trong không gian hai chiều sử dụng hai nhóm đường tròn khác nhau. Số đường tròn của mỗi nhóm và tỷ lệ bán kính của chúng đã được xác định từ trước. Bài toán này là một bài toán quan trọng trong các bài toán cổ điển về bao phủ bởi các đường tròn. Nó thường được nghiên cứu trong trường hợp khi khoảng cách giữa hai điểm là khoảng cách Euclid. Với giả định rằng khoảng cách giữa hai điểm khác nhau được xác định là một đơn vị riêng nào đó trong lĩnh vực logistic, mà gọi chung là khoảng cách phi Euclid. Để giải quyết vấn đề này, các tác giả đề xuất một thuật toán dựa trên sự kết hợp của phương pháp quang - hình học* *sử dụng các nguyên lý Fermat - Huygens và biểu đồ Voronoi. Đặc điểm chính của thuật toán là khả năng giải quyết bài toán bao phủ trong trường hợp khoảng cách không phải Euclid.*

***Từ khóa:*** *bao phủ nhiều lần, các đường tròn khác nhau, phi Euclid, biểu đồ Voronoi, phương pháp quang-hình học*

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